Solving the Teukolsky Equation for a Non-Rotating Black Hole

Sarah Skinner[1] and Peter Diener[2]

[1]Missouri University of Science and Technology
[2]Louisiana State University

Introduction

The overall purpose of this project is to simulate the gravitational waves produced by black holes in binary systems that have high mass ratios. Using traditional methods is too computationally expensive, so instead, we use the Teukolsky equation (shown below) for the smaller black hole. This project's aim is to code the Teukolsky equation[1] for all spin-weights on a non-rotating Schwarzschild background. In the future, we hope to include a Kerr background for all spin weights.

The Teukolsky Equation

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2} - \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{sin^2\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta}+\frac{icos(\theta)}{sin^2\theta}\right]\frac{\partial\psi}{\partial\phi} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - iacos\theta\right]\frac{\partial\psi}{\partial t} + \left[s^2cot^2\theta - s\right]\psi = 4\pi(r^2+a^2cos^2\theta)T$$

$$\Delta = r^2 - 2Mr + a^2$$

- $a \rightarrow$ Kerr spin parameter
- $M \rightarrow$ mass
- $r \rightarrow$ radius
- $\theta \rightarrow$ angle
- $\psi \rightarrow$ field
- $s \rightarrow$ spin-weight
- $T \rightarrow$ matter content

The Teukolsky equation is derived from the perturbed Einstein equations using the Newman-Penrose formalism written in Boyer-Linquist coordinates. The variable s is the spin-weight. The values s=±2 describes the gravitational perturbations, (s=±1) describes the electromagnetic field, and s=0 describes the scalar field.

The variable a is the angular momentum over the mass. With a=0, there is no rotation, for $-M \leq a \leq M$, $a \neq 0$, the black hole has rotation and a Kerr background, and $|a| > M$ is impossible because then it would just be a naked singularity. Because this simulation is going to solve the homogeneous Teukolsky equation first, T is set to zero. Later, the inhomogeneous Teukolsky equation will be implemented in the code.

Construction of Code

The variable a is set to zero in the Teukolsky equation, because this simulation is for a non-rotating black hole. To simplify, the equation was transformed to tortoise coordinates. The fields described by the equation have been decomposed into spin-weighted spherical harmonic modes, so that the simulation can be performed in 1+1 dimensions. To get the desired results, we have to analyze the gravitational waves from an infinite distance away. To solve this, we implemented two layers of hyperboloidal coordinates[2]. Hyperboloidal coordinates compactify the computational domain, so that $\pm\infty$ is mapped to finite coordinate radii.

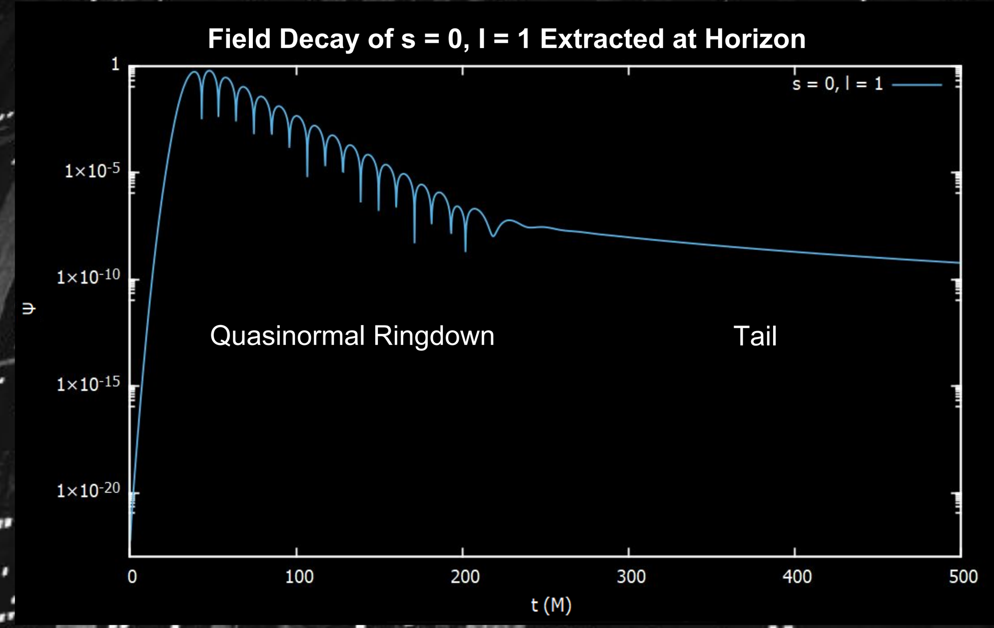
To solve the partial differential equation, nodal discontinuous Galerkin methods[3] are implemented. This method was chosen over other methods due to the need for high accuracy. The error converges exponentially with the order of method when evolving smooth fields. After implementing all of these computational and mathematical methods, the simulation should give us data for the scalar and electromagnetic fields, and gravitational perturbations.

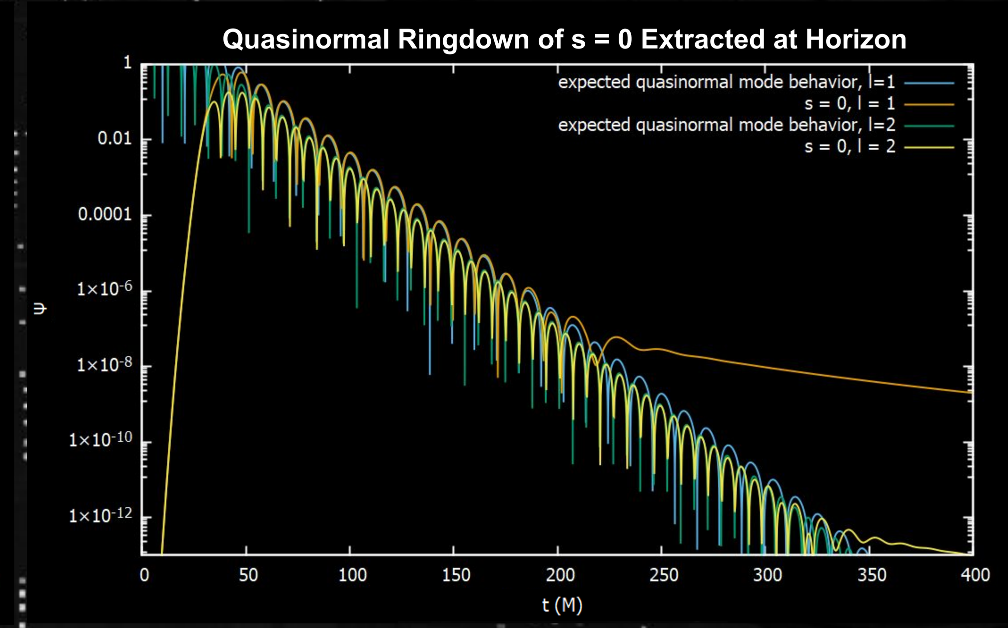
Theoretical Expectations

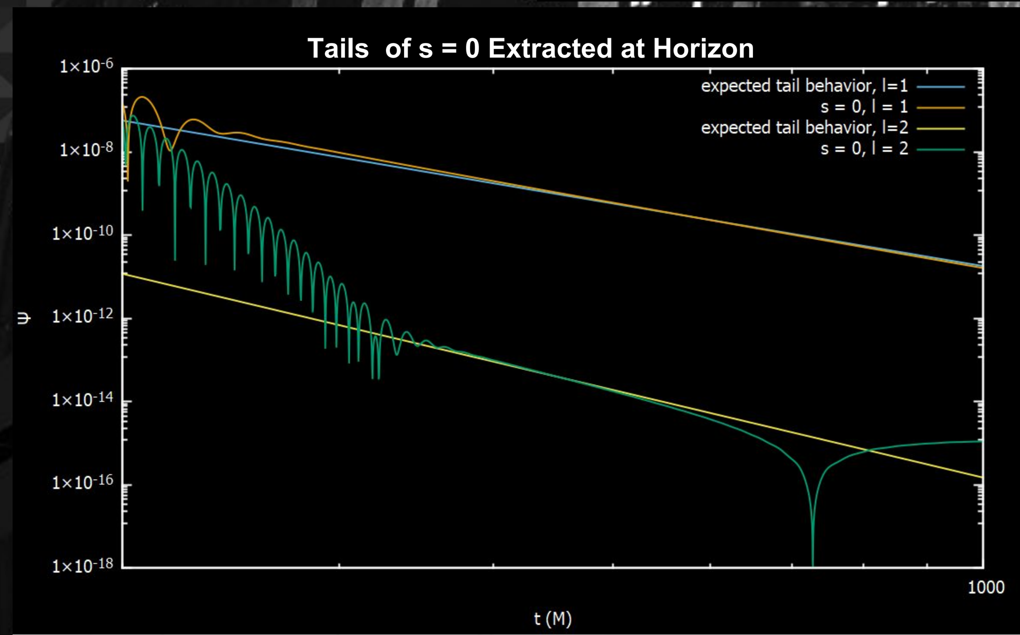
The output from the simulation is ψ (psi), which represents the spin-weighted modes of the field evaluated numerically. The simulation uses geometric units, so time is measured in units of M. The output of the program shows us the decay of the physical field. There are two expected stages of the decay, the quasinormal ringdown and the tails. As we understand from theoretical work, the quasinormal ringdown is determined by the the spacetime curvature close to the black hole and seen when observing the evolution of the field extracted at a fixed distance. It is characterized by exponentially damped sinusoidal oscillations. The tail behavior is caused by the waves backscattering off the spacetime curvature far away from the black hole and is characterized by a power law decay.

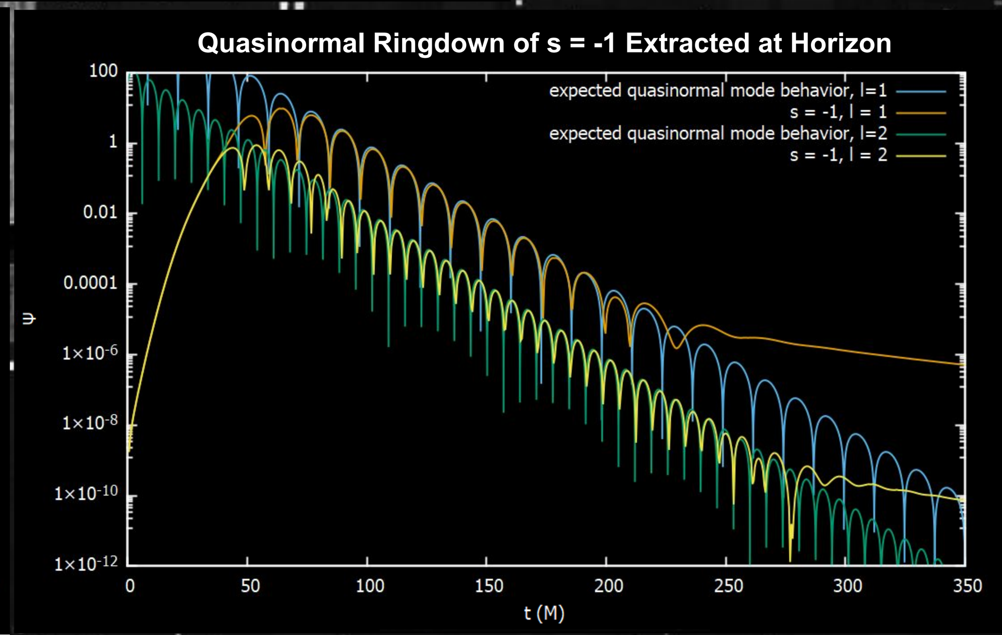
Results

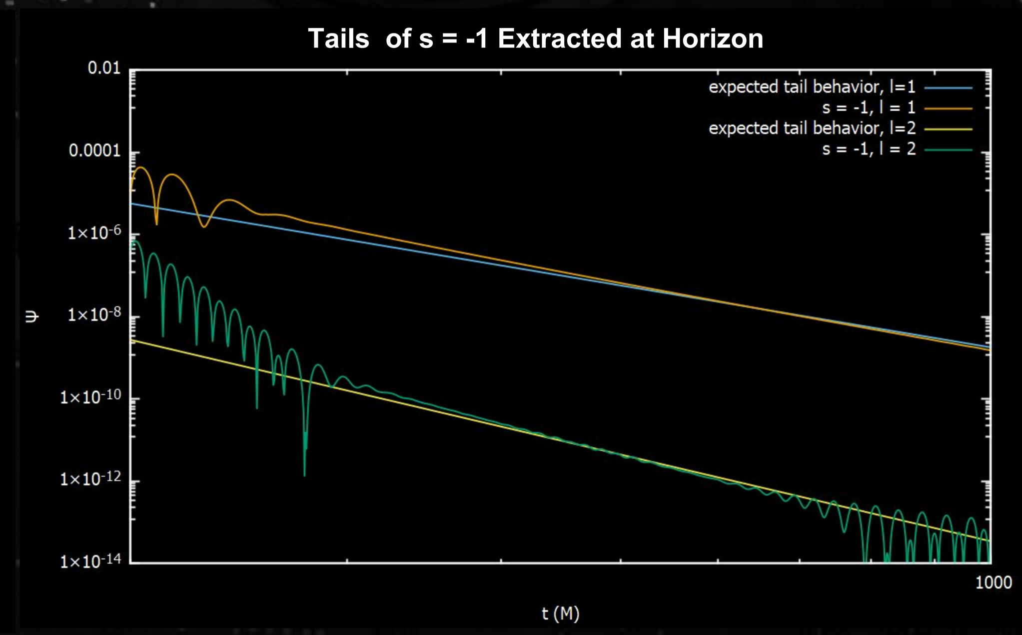
ψ is the spin-weighted field (unitless). We observe ψ at specific radial distances and compare to theoretical work are the horizon and scri+. The horizon is the boundary of the black hole; no waves escape from inside this radius. Scri+ is null infinity (where photons go). The following plots show the fields extracted at these specific locations compared to theoretical calculations.

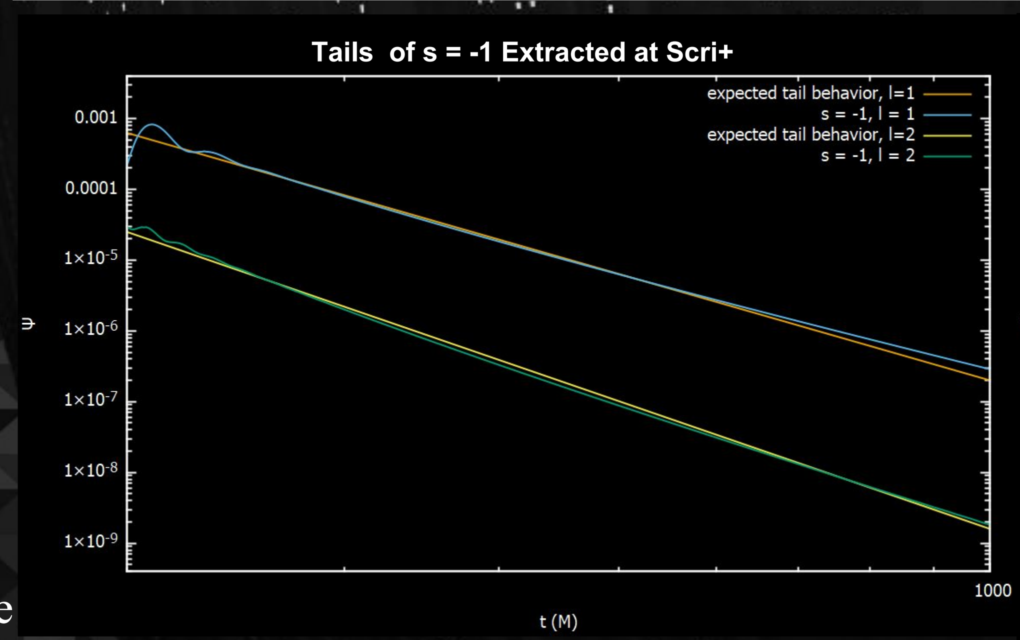


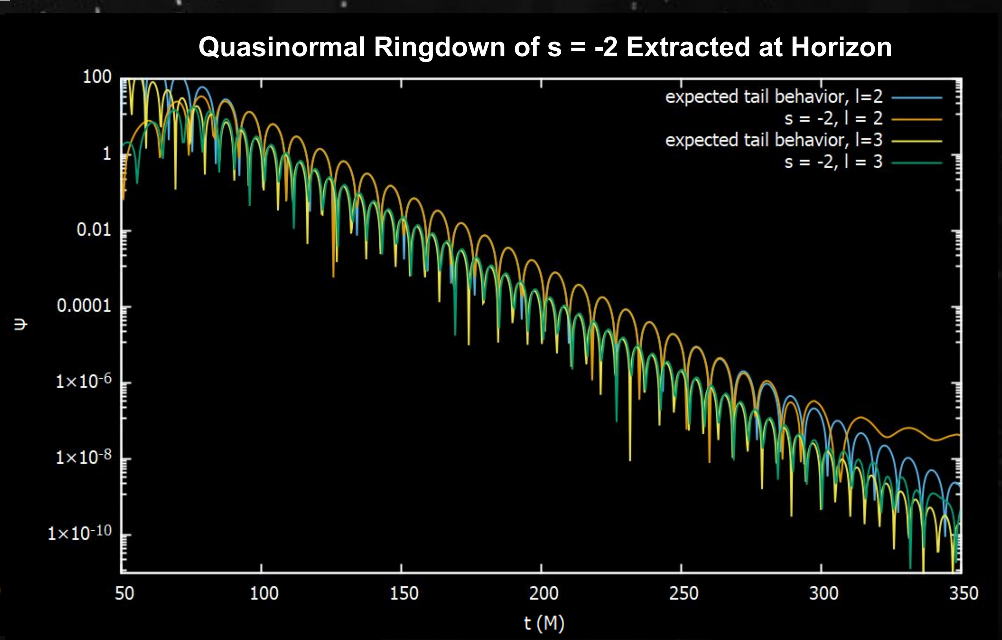


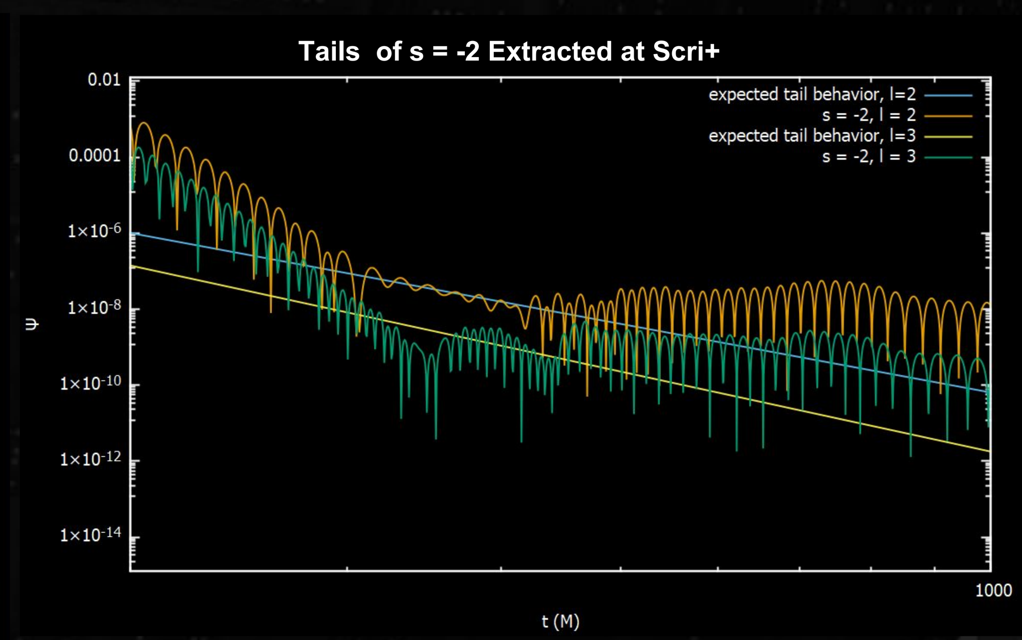












Conclusion

For the spin-weights of 0 and -1 the results are as expected, the quasinormal modes match up with calculated theoretical data provided by Emanuele Berti[4] and the tail behaviors match up with the spin weighted power laws defined by Barack[5] with the notable exception for all l=2 cases. For $l \geq 2$, truncation error increases in later decay. For the spin-weight of -2 the quasinormal modes line up, but we don't see a any clear tail decay due to truncation error even with higher orders. We have yet to resolve this issue and in the future we hope to reduce the amount of error so that we can eventually see a clear tail. For spin-weights of 1 and 2, we do not get any viable data due to an exponential instability. There is not yet a theoretical understanding of this. On the positive side, we can at least produce simulations for at least one spin-weight for the scalar field, electromagnetic field, and gravitational perturbations. In the future, we hope to have a complete generalization of the Teukolsky equation with our code.

References

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